# Differential Geometry III: Riemannian Geometry



**8.1** Let $(X, d_1)$ and $(Y, d_2)$ be two metric spaces and let $f : X \to Y$ be a continuous function. Assume that the following three conditions are satisfied:

1. $f$ is an *open map*, i.e. $f(\mathcal{U})$ is an open subset of $Y$ for any open set $\mathcal{U} \subset X$.
2. $f$ is a *proper map*, i.e. for any $\mathcal{K} \subset Y$ which is compact, $f^{-1}(\mathcal{K})$ is a compact subset of $X$.
3. $Y$ is connected.

Show that $f$ is surjective.

**Solution.** In order to show that $f(X) = Y$, it suffices to show that $f(X)$ is a non-empty, open and closed subset of $Y$. Since $X \neq \emptyset$, we have that $f(X) \neq \emptyset$ as well. Moreover, in view of our assumption that $f$ is an open map, $f(X)$ is an open subset of $Y$. Finally, in order to show that $f(X)$ is closed, we will argue by contradiction: In the case when $f(X) \subseteq Y$ is *not* closed, then the set $C = \text{clos}\big(f(X)\big) \setminus f(X)$ must be non-empty. Let $\bar{y} \in C$; in particular, $y \notin f(X)$. Since $(Y, d_2)$ is a metric space and $f(X)$ is dense in $\text{clos}(f(X))$, there exists a sequence of points $y_n = f(x_n)$ in $f(X)$ with $y_n \xrightarrow{n\to\infty} \bar{y}$. The set $\mathcal{K} = \cup_{n\in\mathbb{N}}\{y_n\} \cup \{\bar{y}\}$ is a compact subset of $Y$ (since, for any open cover $\mathcal{A} = \{\mathcal{U}_i\}_{i\in I}$ of $\mathcal{K}$, any open set $\mathcal{U}_{i_0} \in \mathcal{A}$ with $\bar{y} \in \mathcal{U}_{i_0}$ will contain all but finitely many of the points $y_n$; thus, there exists a finite subset of $\mathcal{A}$ which still covers all of $\mathcal{K}$). By our assumption that $f$ is a proper map, the set $f^{-1}(\mathcal{K}) \subset X$ is compact. Thus the sequence $x_n$ lies inside a compact subset of $X$ and, as a result, has a convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$; let us denote with $\bar{x} = \lim_{k\to\infty} x_{n_k}$. Since $f$ is continuous, we must have

$$f(\bar{x}) = f\big(\lim_{k\to\infty} x_{n_k}\big) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} y_{n_k} = \bar{y}.$$

Thus, $\bar{y} \in f(X)$, which is a contradiction.

**8.2** Let $(\mathcal{M}, g)$ be a *homogeneous* Riemannian manifold (i.e. for any two points $p, q \in \mathcal{M}$, there exists an isometry $F : \mathcal{M} \to \mathcal{M}$ such that $F(p) = q$). Show that $(\mathcal{M}, g)$ is complete.

*Hint: You might want to use the fact that isometries map geodesics to geodesics, to infer that every point on $\mathcal{M}$ has the same injectivity radius. Under this condition, can a maximal geodesic be incomplete?*

**Solution.** In view of the Hopf–Rinow theorem, it suffices to show that any maximally extended geodesic $\gamma : I \to \mathcal{M}$ is complete, i.e. $I = \mathbb{R}$. To this end, it suffices to show that there exists an $\epsilon > 0$ such that the injectivity radius of any point $p \in \mathcal{M}$ satisfies $\iota(p) > \epsilon$. Assume for a moment that this is true; in that case, let $\gamma : (a, b) \to \mathcal{M}$ be a geodesic of $(\mathcal{M}, g)$ parametrised with *unit* speed (i.e. $\|\dot{\gamma}\| = 1$) and such that $b < +\infty$. Let $s_0 = \frac{1}{2}(b - \epsilon)$ and $q = \gamma(s_0)$. Then, since $\iota(q) > \epsilon$, any geodesic starting from $q$ with initial velocity $v \in T_q\mathcal{M}$ with $\|v\| = 1$ can be defined for time $t < \epsilon$ (since $\exp_q(tv)$ is well defined for $s < \epsilon$). Therefore, the geodesic $\gamma(s)$ can be extended for $s \in (a, s_0 + \epsilon) = (a, b + \frac{\epsilon}{2})$. In the case when $a > -\infty$, by arguing similarly near the endpoint $s = a$, we also deduce that $\gamma(s)$ can be extended as a geodesic for $s \in (a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})$. As a result, if $\gamma$ is maximally extended, then it is necessary that $a = -\infty$ and $b = +\infty$, i.e. that $\gamma$ is complete.

It remains to prove that there exists an $\epsilon > 0$ such that $\iota(p) > \epsilon$ for all $p \in \mathcal{M}$. In fact, we will prove the stronger statement that every point on $\mathcal{M}$ has the same injectivity radius (which is, thus, a strictly positive number). Let $p, q \in \mathcal{M}$; our assumption that $(\mathcal{M}, g)$ is homogeneous means that there exists an isometry $F : \mathcal{M} \to \mathcal{M}$ such that $F(p) = q$. In view of Exercise 6.1, the isometry $F$ "commutes" with the exponential map, i.e.

$$F \circ \exp_p = \exp_{F(p)} \circ dF_p = \exp_q \circ dF_p.$$

Therefore, let $\rho > 0$ be such that $\exp_p$ is well defined on the ball

$$B_\rho^{(p)} \doteq \{v \in T_p\mathcal{M} : \|v\| \leqslant \rho\}$$

and $\exp_p : B_\rho^{(p)} \to \exp_p(B_\rho^{(p)}) \subset \mathcal{M}$ is a diffeomorphism. Then, since $\exp_q(dF(v)) = F(\exp_p(v))$ and $F$ is a diffeomorphism (as an isometry), the map $\exp_q(w)$ is well defined for any $w \in dF(B_\rho^{(p)})$ and is a diffeomorphism on $dF(B_\rho^{(p)})$. Since $\|dF(v)\| = \|v\|$ (by our assumption that $F$ is an isometry), we have that

$$dF(B_\rho^{(p)}) = B_\rho^{(q)} \doteq \{w \in T_q\mathcal{M} : \|w\| \leqslant \rho\}.$$

Recall that the injectivity radius $\iota(z)$ of $z \in \mathcal{M}$ is defined as

$$\iota(z) = \sup\Big\{\rho > 0 : \exp_z(\cdot) \text{ is well-defined on } B_\rho^{(z)} \text{ and } \exp_z : B_\rho^{(z)} \to \mathcal{M} \text{ is a diffeomorphism on its image}\Big\}.$$

Therefore, the above discussion implies that $\iota(p) = \iota(q)$.

**8.3** A Riemannian manifold $(\mathcal{M}, g)$ is called *isotropic* if, for every $p \in \mathcal{M}$ and every $v_1, v_2 \in T_p\mathcal{M}$ with $\|v_1\| = \|v_2\|$, there exists an isometry $F$ of $(\mathcal{M}, g)$ with $F(p) = p$ and $dF(v_1) = v_2$ (in other words, around $p$ the space $(\mathcal{M}, g)$ "looks the same" in every direction).

(a) Can you find an example of an isotropic Riemannian manifold? An example of a homogeneous but not isotropic Riemannian manifold?

(b) Show that a connected, complete and isotropic Riemannian manifold $(\mathcal{M}, g)$ is also homogeneous (see Ex. 8.2). *Hint: For any $p, q \in \mathcal{M}$, let $x$ be the midpoint of a geodesic segment connecting $p$ to $q$ (why does it exist?); consider the set of isometries that fix $x$.*

(c) Show that the assumption on $(\mathcal{M}, g)$ being complete above is redundant, i.e. that every connected and isotropic Riemannian manifold is also complete (and, hence, homogeneous). *Hint: If $\gamma : (a, b) \to \mathcal{M}$, $0 \in (a, b)$, is a maximal geodesic, show that the isotropic condition implies that $a = -b$. Note that this should be true for all points along the geodesic.*

**Solution.** (a) The sphere, the plane and the hyperbolic plane are all examples of homogeneous and isotropic Riemannian manifolds. An example of a non-compact homoegeneous but not isotropic space is the cylinder $\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ equipped with the induced metric $g$ from $(\mathbb{R}^3, g_E)$. If we use polar coordinates $(r, \theta)$ in the $(x, y)$-plane (i.e. $x = r\cos\theta$, $y = r\sin\theta$), then $\mathcal{C}$ corresponds to the subset $\{r = 1\}$ of $\mathbb{R}^3$ parametrized by $(\theta, z) \in [0, 2\pi) \times \mathbb{R}$; in these coordinates,

$$g = d\theta^2 + dz^2$$

(note, in particular, that $(\mathcal{C}, g)$ is locally flat). Translations in the $z$ direction and rotations in the $(x, y)$ plane (i.e. translations in the $\theta$ variable) are all isometries of $\mathcal{C}$, and a combination of those can map any pair of points $p, q \in \mathcal{C}$ to each other. However, $(\mathcal{C}, g)$ cannot be isotropic: Any isometry $F$ will have to map geodesics to geodesics and, therefore, if an isometry $F : \mathcal{C} \to \mathcal{C}$ existed which fixed a point $p \in \mathcal{C}$ and $dF|_p$ mapped $\frac{\partial}{\partial\theta}\Big|_p$ to a tangent vector parallel to $\frac{\partial}{\partial z}\Big|_p$, then $F$ should map the maximal geodesic through $p$ in the direction of $\frac{\partial}{\partial\theta}\Big|_p$ to the maximal geodesic through $\frac{\partial}{\partial z}\Big|_p$; however, this is impossible, since the former geodesic has compact image (being a closed circle), while the latter is an infinite straight line.

An example of a compact homogeneous but not isotropic space is the flat torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ (i.e. where points $(x, y)$ and $(\bar{x}, \bar{y})$ in $\mathbb{R}^2$ are identified if $(x-\bar{x}, y-\bar{y}) \in \mathbb{Z}\times\mathbb{Z}$; thus, $\mathbb{T}^2$ is parametrized by the unit square $[0, 1) \times [0, 1)$), equipped with the flat metric $g_E = dx^2 + dy^2$. In this case, any translation $(x, y) \to (x + a_1, y + a_2) \mod \mathbb{Z}^2$ for some fixed $(a_1, a_2) \in \mathbb{R}^2$ is an isometry of $(\mathbb{T}^2, g_E)$; thus, any pair of points $p, q \in \mathbb{T}^2$ can be mapped to each other through an isometry. On the other hand, there can be no isometry fixing $(0, 0) \mod \mathbb{Z}^2$ and mapping $\frac{\partial}{\partial x}$ to a tangent vector parallel to $\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$; this is because the corresponding closed geodesics of $(\mathbb{T}^2, g_E)$ in these directions (namely the bottom edge of the unit square and the diagonal of the square, respectively) have different length (1 and $\sqrt{2}$, respectively) and, thus, one cannot be the image of the other through an isometry.

(b) Let $p, q$ be two points on $\mathcal{M}$. Since $(\mathcal{M}, g)$ was assumed to be complete, the theorem of Hopf–Rinow states that the map $\exp_p : T_p\mathcal{M} \to \mathcal{M}$ is onto. Therefore, there exists a geodesic $\gamma : [0, 1] \to \mathcal{M}$ such that $\gamma(0) = p$ and $\gamma(1) = q$ (this is simply the geodesic $\gamma(t) = \exp_p(tv)$ for a $v \in T_p\mathcal{M}$ such that $\exp_p v = q$). Let us consider the point $z = \gamma(\frac{1}{2})$ and let us set $w = -\dot{\gamma}(\frac{1}{2}) \in T_z\mathcal{M}$. Since $(\mathcal{M}, g)$ was assumed to be isotropic, there exists an isometry $F : \mathcal{M} \to \mathcal{M}$ such that

$$F(z) = z \quad \text{and} \quad dF|_z(w) = -w. \tag{1}$$

We will show that $F$ maps $\gamma([0, 1])$ to itself by switching the endpoints $\gamma(0) = p$ and $\gamma(1) = q$.

Let us consider the curves $\tilde{\gamma}(t) = \exp_z(tw)$ and $\tilde{\gamma}'(t) = \exp_z(-tw)$, $t \in [-\frac{1}{2}, \frac{1}{2}]$. Note that $\tilde{\gamma}(t) = \gamma(-t + \frac{1}{2})$ and $\tilde{\gamma}'(t) = \gamma(t + \frac{1}{2})$ (this can be readily verified by checking that all those curves are geodesics and the corresponding pairs have the same initial conditions at $t = 0$, hence they are equal by the uniqueness of solutions to the geodesic equation). Therefore,

$$\tilde{\gamma}(\frac{1}{2}) = p, \quad \tilde{\gamma}'(\frac{1}{2}) = q.$$

Using the fact that any isometry "commutes" with the exponential map (see Ex. 6.1), we obtain

$$F(\exp_z(tw)) = \exp_{F(z)}(t \cdot dF|_z(w)) \quad \text{for } t \in [-\frac{1}{2}, \frac{1}{2}].$$

Using the property (1) of $F$ and the definition of the curves $\tilde{\gamma}$, $\tilde{\gamma}'$, we infer that

$$F(\tilde{\gamma}(t)) = F(\exp_z(tw)) = \exp_{F(z)}(tdF|_z(w)) = \exp_z(-tw) = \tilde{\gamma}'(t)$$

and, therefore, setting $t = \frac{1}{2}$:

$$F(p) = q.$$

(c) In view of the Hopf-Rinow theorem, in order to show that $(\mathcal{M}, g)$ is complete it suffices to show that it is geodesically complete. To this end, we will argue by contradiction: Assume that $(\mathcal{M}, g)$ is not geodesically complete, then there exists a maximally extended geodesic $\gamma : (a, b) \to \mathcal{M}$ such that $a > -\infty$ or $b < +\infty$; without loss of generality, let us assume that $b < +\infty$ (the case $a > -\infty$ being completely analogous). The fact that $\gamma$ is maximally extended means that there exists no other geodesic $\tilde{\gamma} : I \to \mathcal{M}$ on an interval $I \supsetneq (a, b)$ such that $\tilde{\gamma}|_{(a,b)} = \gamma$.

Let $t_0 \in (a, b)$ be sufficiently close to $b$ so that

$$t_0 - a > b - t_0$$

(the left hand side being $+\infty$ in the case when $a = -\infty$) and let us set $z = \gamma(t_0)$ and $w = \dot{\gamma}(t_0) \in T_z\mathcal{M}$. By our assumption that $(\mathcal{M}, g)$ is isotropic, there exists an isometry $F : \mathcal{M} \to \mathcal{M}$ such that

$$F(z) = z \quad \text{and} \quad dF|_z(w) = -w. \tag{2}$$

We will show that $F$ "flips" $\gamma$ around $z$, thus allowing us to extend $\gamma(t)$ beyond $t = t_0$ for time $t = t_0 + (t_0 - a) > t_0 + (b - t_0) = b$.

Let us consider the curve $\tilde{\gamma}(s) = \exp_z(sw)$. Note that $\tilde{\gamma}(s) = \gamma(s + t_0)$ (since, as before, both curves are geodesics and have the same initial conditions at $s = 0$). Thus, since $\gamma(t)$ is well-defined for $t \in (a, b)$, the curve $\tilde{\gamma}(s)$ is well-defined for $s \in (a - t_0, b - t_0)$. Moreover, we have that

$$F(\tilde{\gamma}(-s)) = \tilde{\gamma}(s) \quad \text{for } s \in (-\delta, \delta)$$

for any $0 < \delta < \min\{t_0 - a, b - t_0\} = b - t_0$, in view of the fact that both $F(\tilde{\gamma}(-s))$ and $\tilde{\gamma}(s)$ are geodesics (since $F$ maps geodesics to geodesics) and they satisfy $F(\tilde{\gamma}(0)) = \tilde{\gamma}(0)$ and $\frac{d}{ds}F(\tilde{\gamma}(-s))\Big|_{s=0} = dF(-w) = w = \frac{d}{ds}\tilde{\gamma}(-s)\Big|_{s=0}$.

Let us consider now the curve $\hat{\gamma} : (a - t_0, t_0 - a) \to \mathcal{M}$ defined by

$$\hat{\gamma}(s) = \begin{cases} \tilde{\gamma}(s), & s \in (a - t_0, 0], \\ F(\tilde{\gamma}(-s)), & s \in [0, t_0 - a). \end{cases}$$

This is a $C^1$ curve (since, as explained above, $F(\tilde{\gamma}(0)) = \tilde{\gamma}(0)$ and $\frac{d}{ds}F(\tilde{\gamma}(-s))\Big|_{s=0} = \frac{d}{ds}\tilde{\gamma}(-s)\Big|_{s=0}$), satisfying in addition the following properties:

- $\hat{\gamma}(s)$ satisfies the geodesic equation on $(a - t_0, 0) \cup (0, t_0 - a)$, since it coincides there with the geodesics $\tilde{\gamma}(s)$ and $F(\tilde{\gamma}(-s))$, respectively.
- $\hat{\gamma}(s) = \tilde{\gamma}(s)$ on $(-\delta, \delta)$ for any $0 < \delta < b - t_0$ as explained above and, therefore, $\hat{\gamma}(s)$ also satisfies the geodesic equation around $s = 0$.

Thus, $\hat{\gamma}$ is a geodesic. Since $\hat{\gamma}(s) = \tilde{\gamma}(s)$ in a neighborhood of $s = 0$, we deduce from the uniqueness property of the initial value problem for the geodesic equation that $\hat{\gamma}(s) = \tilde{\gamma}(s)$ for $s$ in the domain of definition of both $\hat{\gamma}$ and $\tilde{\gamma}$; equivalently,

$$\hat{\gamma}(t - t_0) = \gamma(t) \quad \text{for } t \in (a, b).$$

Therefore, $\hat{\gamma}(t - t_0)$ is a geodesic defined on $(a, 2t_0 - a) \supsetneq (a, b)$ which coincides with $\gamma(t)$ on $(a, b)$, which is a contradiction in view of our assumption that $\gamma$ is maximally extended.

**8.4** Let $F : (\mathcal{M}, g) \to (\mathcal{N}, h)$ be a local isometry between two Riemannian manifolds (recall that a local isometry is a map for which $dF|_p : T_p\mathcal{M} \to T_{F(p)}\mathcal{N}$ is 1-1 and onto for all $p \in \mathcal{M}$ and $F^*h = g$). Assume that $\mathcal{N}$ is connected and $(\mathcal{M}, g)$ is complete. Show that $F$ is onto and that $(\mathcal{N}, h)$ is also complete. Is $F$ necessarily $1 - 1$?

**Solution.** Let $p$ be a point in $\mathcal{M}$. Without loss of generality, we can assume that $\mathcal{M}$ is connected (otherwise we can just restrict to the connected component of $p$ in $\mathcal{M}$). Since $(\mathcal{M}, g)$ is complete, by the Hopf–Rinow theorem we know that it is also geodesically complete, hence $\exp_p$ is defined on all of $T_p\mathcal{M}$ and $\exp_p(T_p\mathcal{M}) = \mathcal{M}$. For any $v \in T_p\mathcal{M}$, we have

$$F(\exp_p v) = \exp_{F(p)}(dF|_p(v)) \tag{3}$$

(the proof of the above is the same as in the case when $F$ is a global isometry, since it is only local in nature), where the exponential map in the right hand side is associated to the Riemannian metric $h$ on $\mathcal{N}$. Therefore, $\exp_{F(p)}(w)$ is well-defined for any $w \in T_{F(p)}\mathcal{N}$ of the form $w = dF|_p(v)$, $v \in T_p\mathcal{M}$. Since $dF|_p : T_p\mathcal{M} \to T_{F(p)}\mathcal{N}$ is onto (due to our assumption that $F$ is a local isometry), we infer that $\exp_{F(p)}$ is defined on all of $T_{F(p)}\mathcal{N}$, i.e. every geodesic of $(\mathcal{N}, h)$ passing through $F(p)$ is complete. By the Hopf–Rinow theorem (in view of the fact that $\mathcal{N}$ was assumed to be connected), this implies that $(\mathcal{N}, h)$ is complete and that $\exp_{F(p)} : T_{F(p)}\mathcal{N} \to \mathcal{N}$ is onto. Since, as explained earlier, $dF|_p : T_p\mathcal{M} \to T_{F(p)}\mathcal{N}$ is onto as well, we deduce that $\exp_{F(p)} \circ dF|_p : T_p\mathcal{M} \to \mathcal{N}$ is onto; in view of the relation (3), this implies that, for every $y \in \mathcal{N}$, there exists a $v \in T_p\mathcal{M}$ such that, setting $x = \exp_p v \in \mathcal{M}$:

$$F(x) = y$$

i.e. that $F(\mathcal{M}) = \mathcal{N}$.

The map $F$ doesn't necessarily have to be 1-1: For instance, the quotient map $F : (\mathbb{R}^2, g_E) \to (\mathbb{T}^2, g_E)$ defined by $F(x, y) = (x \mod \mathbb{Z},\ y \mod \mathbb{Z})$ satisfies all of the conditions of the exercise but is not 1-1.